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OPTIMAL DESIGN OF RIGID-PLASTIC SIMPLY SUPPORTED BEAMS UNDER IMPULSIVE LOADING

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Abstract—Optimal design of a rigid-plastic stepped beam is being discussed. Such beam dimensions are sought on which the beam of constant volume attains a minimum local or mean deflection. The beam is subjected to a constant initial velocity field. An exact solution to the problem is found. This is compared with four types of mode form solutions. The error made by using the mode form solutions is estimated. Some suggestions for an optimal design beams are made.

1. INTRODUCTION

In previous papers by Lepik and Mróz[1, 2], the problem of optimal design of rigid-plastic stepped beams under impulsive loading was discussed. It was assumed, that at the initial instant the beam attains, through impulsive loading, the kinetic energy κ_0 and the subsequent motion proceeds to the modal form. By solving this problem some new aspects, such as nonuniqueness and instability of some modes, appeared. It was not quite clear either these effects were caused by the character of the problem considered or are they due to the approximate method of the mode form solutions. To answer this question an exact solution of some problems is needed. From the mathematical point of view this is certainly more complicated, since the effect of moving plastic hinges must be taken into account.

The main purpose of this paper is to obtain an exact solution for a stepped simply supported beam and to compare it with mode form solutions. It is assumed that the beam is subjected to an impulsive load which imparts a constant initial velocity to all sections. The basic differential equations are given in Section 2. It follows from the integration of these equations that three different cases exist. They are examined in Sections 3–5. On the basis of these results the solution of the optimization problem is given in Section 6. Some possibilities for getting mode form solutions are discussed in Sections 7 and 8. The exactness of results obtained by using these solutions is estimated. In the final Section some practical suggestions for the optimal design of rigid-plastic structures will be made.

2. BASIC EQUATIONS

Let us consider a rigid-plastic beam with rectangular cross-section and segmentwise constant thickness, simply supported on both edges (Fig. 1). The beam is under impulsive loading; it is assumed that a constant initial transverse velocity field v_0^* is prescribed.

The equations of motion are

$$\frac{\partial M^*}{\partial x} = Q^*, \quad \frac{\partial Q^*}{\partial x} = \rho B h^*(x) \frac{\partial^2 w^*}{\partial t^2}, \tag{1}$$

where M^* and Q^* denote the bending moment and the shear force, ρ is the density, B, h^* and w^* respectively stand for the width, height and deflection of the beam.

In the following part only designs of the same volume will be considered. By introducing the

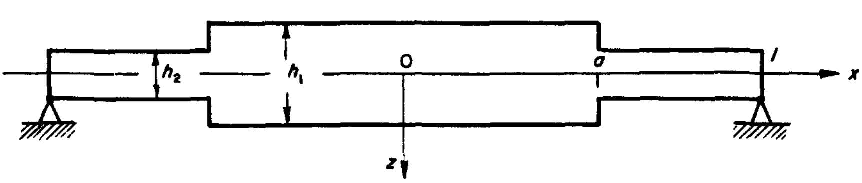

Fig. 1. Beam dimensions.

quantities $\Delta = \alpha\gamma + 1 - \alpha$ and $\gamma = h_1/h_2$ the beam's volume can be put into the form $V = 2\,Bh_2l/\Delta$. The thickness h_1 and h_2 are

$$h_1 = \frac{\gamma V}{2Bl\Delta}, \qquad h_2 = \frac{V}{2Bl\Delta}. \tag{2}$$

For convenience the following non-dimensional quantities are introduced:

$$\xi = \frac{x}{l}, \quad \alpha = \frac{a}{l}, \quad h = \frac{h^*}{h_2}, \quad \tau = \frac{3\delta_0 Vt}{8\rho Bv_0 l^3},$$

$$w = \frac{3\delta_0 Vw^*}{16\rho Bv_0{}^2 l^3}; \quad M = \frac{4M^*}{\delta_0 Bh_2{}^2}, \quad Q = \frac{4lQ^*}{\delta_0 Bh_2{}^2}, \tag{3}$$

where δ_0 is the yield stress.

Henceforth dots will denote differentation with respect to the non-dimensional time τ. Now the eqns (1) take the form

$$\frac{\partial M}{\partial \xi} = Q, \quad \frac{\partial Q}{\partial \xi} = 6\Delta h(\xi)\ddot{w}. \tag{4}$$

In case of a stepped beam the following conditions must be fulfilled: (i) $h(\xi) = \gamma$, $|M| \leqslant \gamma^2$ for $\xi \in [0, \alpha]$; (ii) $h(\xi) = 1$, $|M| \leqslant 1$ for $\xi \in [\alpha, 1]$.

The actual velocity $\partial w^*/\partial t$ and the non-dimensional velocity $\dot{w}$ are connected by the formula

$$\frac{\partial w^*}{\partial t} = 2v_0{}^*\dot{w} \tag{5}$$

and, consequently $\dot{w}(\xi, 0) \equiv 0.5$. When $\tau = 0$ a plastic hinge appears in the section $\xi = 1$. This hinge begins to travel towards to the centre of the beam $\xi = 0$. Let us denote its location by the quantity $\beta = \beta(\tau)$. The non-dimensional velocity field can be presented in the form

$$\dot{w} = \begin{cases} 0.5 & \text{for} \quad \xi \in [0, \beta], \\ 0.5\dfrac{1-\xi}{1-\beta} & \text{for} \quad \xi \in [\beta, 1]. \end{cases} \tag{6}$$

Now we shall differentiate the eqns (6) with respect to τ. Putting this result into (4), integrating this system and taking account of the conditions $M(\beta) = 1$ and $Q(\beta) = 0$ we obtain

$$M = \begin{cases} 1 & \text{for} \quad \xi \in [0, \beta], \\ 1 + \dfrac{\Delta\dot{\beta}}{2(1-\beta)^2}(3 - 2\beta - \xi)(\xi - \beta)^2 & \text{for} \quad \xi \in [\beta, 1]. \end{cases} \tag{7}$$

The boundary condition $M(1) = 0$ gives $\Delta(1-\beta)\dot{\beta} = -1$. This differential equation has the integral

$$1 - \beta(\tau) = \sqrt{\left(\frac{2\tau}{\Delta}\right)}. \tag{8}$$

This formula is valid only for $\beta \geqslant \alpha$. When the hinge has moved to the section $\xi = \alpha$, this phase of motion comes to an end. It follows form the condition $\beta(\tau_1) = \alpha$ that

$$\tau_1 = 0.5\Delta(1-\alpha)^2. \tag{9}$$

Now we have to determine the residual deflections for the instant $\tau = \tau_1$. For $\xi \geqslant \beta$ we obtain

$$\mathrm{d}w = \frac{1-\xi}{2(1-\beta)}\,\mathrm{d}\tau = -0.5\Delta(1-\xi)\,\mathrm{d}\beta.$$

By integrating this equation we get

$$w = -0.5\Delta(1-\xi)\beta + \varphi(\xi).$$

The function $\varphi(\xi)$ can be calculated from the condition $w(\beta) = 0.5\tau$. Having done this we see that

$$w = 0.25\Delta(1-\xi)(1+\xi-2\beta)$$

For $\xi < \beta$ the equation $w = 0.25\Delta(1-\alpha)^2$ holds good.

For the instant $\tau = \tau_1$, where $\beta(\tau_1) = \alpha$, these results take the form

$$w_1(\xi) = w(\xi, \tau_1) = \begin{cases} 0.25\Delta(1-\alpha)^2 & \text{for} \quad \xi \in [0, \alpha], \\ 0.25\Delta(1-\xi)(1+\xi-2\alpha) & \text{for} \quad \xi \in [\alpha, 1]. \end{cases} \tag{10}$$

It follows from eqns (7)–(8) that the inequality $|M| \leq 1$ is always fulfilled, consequently, this phase of motion takes place for arbitrary values of the parameters $0 < \alpha < 1$ and $\gamma > 1$. The following phases of motion essentially depend upon the values α and γ. Here three different cases, which are examined in Sections 3–5, can be distinguished.

3. FIRST CASE

When the relation $\gamma = h_1/h_2$ is great enough, a stationary plastic hinge at $\xi = \alpha$ becomes evident in the second phase of motion. The deflection rate field can be revealed in the form

$$\dot{w} = \begin{cases} v(t) & \text{for} \quad \xi \in [0, \alpha], \\ v(t)\dfrac{1-\xi}{1-\alpha} & \text{for} \quad \xi \in [\alpha, 1]. \end{cases} \tag{11}$$

Again we shall differentiate these equations with respect to time τ. Transferring this result into (4), integrating these equations and making use of the conditions $M(1) = 0$, $Q(0) = 0$, $M(\alpha-) = 1$, $Q(\alpha-) = Q(\alpha+)$, we get

$$M = \begin{cases} 1 - 3\gamma\Delta\dot{v}(\alpha^2 - \xi^2) & \text{for} \quad \xi \in [0, \alpha], \\ \dfrac{\Delta\dot{v}}{1-\alpha}[-6\alpha\gamma(1-\alpha)(1-\xi) - 2 + 6\alpha - 3\alpha^2 - 3\alpha\xi(2-\alpha) + 3\xi^2 - \xi^3] & \text{for} \quad \xi \in [\alpha, 1]. \end{cases} \tag{12}$$

The continuity condition $M(\alpha-) = M(\alpha+)$ gives

$$\dot{v} = -\frac{1}{2\Delta(1-\alpha)(3\alpha\gamma + 1 - \alpha)}. \tag{13}$$

This equation has the integral

$$v(\tau) = 0.5 - \frac{\tau - \tau_1}{2\Delta(1-\alpha)(3\alpha\gamma + 1 - \alpha)}. \tag{14}$$

Here τ_1 is the duration of the first phase, which is calculated from the formula (9). The motion stops at the instant $\tau = \tau_f$, where $v(\tau_f) = 0$. Calculating the whole response time from eqn (14), we obtain

$$\tau_f = \tau_1 + \Delta(1-\alpha)(3\alpha\gamma + 1 - \alpha). \tag{15}$$

Now we have to integrate the eqn (14) in the interval $[\tau_1, \tau_f]$; making use of (11) we get formulas for the residual deflections, obtained in the second phase of motion:

$$w_2(\xi) = \begin{cases} 0.25\Delta(1-\alpha)(3\alpha\gamma+1-\alpha) & \text{for } \xi \in [0,\alpha], \\ 0.25\Delta(1-\xi)(3\alpha\gamma+1-\alpha) & \text{for } \xi \in [\alpha,1]. \end{cases} \tag{16}$$

The case above will be realized, if $M(0) \leq \gamma^2$. This condition leads to the inequality

$$3\gamma\alpha^2 \leq 2(1-\alpha)(\gamma^2-1)(3\alpha\gamma+1-\alpha). \tag{17}$$

On the basis of (13) we have $\dot{v} < 0$ and also $\ddot{w} < 0$. According to (4) we have $\partial Q/\partial\xi < 0$. Since $Q(0) = 0$, it must be $Q = \partial M/\partial\xi < 0$ and $M(\xi)$ is a decreasing function. Hence it follows that the inequalities $|M| \leq \gamma^2$ for $\xi \in [0,\alpha]$ and $|M| \leq 1$ for $\xi \in [\alpha, 1]$ are fullfilled and consequently our solution is valid, if only the inequality (17) holds good.

4. SECOND CASE

Here, besides the first phase of motion, which was considered in Section 2, two more phases occur. In the second phase we have stationary hinges at $\xi = 0$ and $\xi = \alpha$. When $\tau = \tau_2$ the hinge at $\xi = \alpha$ disappears and the motion in the third phase proceeds with a hinge in the centre of the beam. Let us analyse these phases in detail.

The second phase

Here we shall consider the following yield mechanism

$$\dot{w} = \begin{cases} (1-\alpha)\dot{\varphi} + (\alpha-\xi)\dot{\psi} & \text{for } \xi \in [0,\alpha], \\ (1-\xi)\dot{\varphi} & \text{for } \xi \in [\alpha,1]. \end{cases} \tag{18}$$

The meaning of the quantities $\dot{\varphi}$ and $\dot{\psi}$ becomes evident from Fig. 2.

Integrating the eqns (4) and satisfying the conditions $Q(0) = 0$, $Q(\alpha-) = Q(\alpha+)$ we obtain

$$M = \begin{cases} \Delta\gamma[3(1-\alpha)\xi^2\ddot{\varphi} + (3\alpha-\xi)\xi^2\ddot{\psi}] + C_1 & \text{for } \xi \in [0,\alpha], \\ \Delta\{[3(\xi-\alpha)^2 - \xi^3 + 3\alpha^2\xi + 6\alpha\gamma(1-\alpha)\xi]\ddot{\varphi} + 3\alpha^2\gamma\xi\ddot{\psi}\} + C_2 & \text{for } \xi \in [\alpha,1]. \end{cases}$$

The constants of integration C_1, C_2 and the quantities $\ddot{\varphi}$, $\ddot{\psi}$ will be calculated from the conditions $M(0) = \gamma^2$, $M(\alpha) = 1$, $M(1) = 0$, $M(\alpha-) = M(\alpha+)$. Having carried out these calculations we get

$$\ddot{\varphi} = \frac{3(\gamma^2-1)(1-\alpha)-2\alpha}{\Delta(1-\alpha)^2\alpha[4(1-\alpha)+3\alpha\gamma]}, \qquad \ddot{\psi} = \frac{3\gamma\alpha^2 - 2(\gamma^2-1)(1-\alpha)(1-\alpha+3\alpha\gamma)}{\Delta\gamma\alpha^3(1-\alpha)[4(1-\alpha)+3\alpha\gamma]}. \tag{19}$$

This solution will be valid, if the following two conditions are observed:

(i) $\dot{\psi}$ is an increasing function and consequently $\ddot{\psi} > 0$;

(ii) the bending moment M does not have extremum inside the interval $\xi \in [0, \alpha]$; this requirement will be met if $M''(0) > 0$.

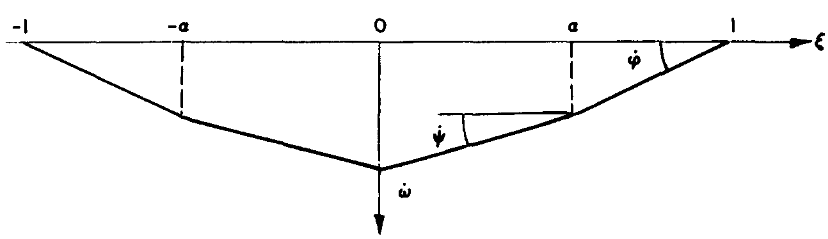

Fig. 2. Yield mechanism with three plastic hinges.

Making use of the formulas (18)–(19), the conditions $\ddot{\psi} > 0$ and $M''(0) > 0$ can be presented in the form

$$3\gamma\alpha^2 > 2(1-\alpha)(\gamma^2-1)(3\alpha\gamma+1-\alpha),$$
$$\gamma\alpha^2 < (1-\alpha)(\gamma^2-1)[3\alpha\gamma+2(1-\alpha)]. \tag{20}$$

The hinge at $\xi = \alpha$ disappears at the instant $\tau = \tau_2$ for which

$$\ddot{\psi}(\tau_2-\tau_1) = \dot{\varphi}(\tau_1) + \ddot{\varphi}(\tau_2-\tau_1). \tag{21}$$

The quantity $\dot{\varphi}(\tau_1)$ can be calculated from (6) taking $\beta = \alpha$; this gives $\dot{\varphi}(\tau_1) = [2(1-\alpha)]^{-1}$ and thus

$$\tau_2 = \tau_1 + \frac{1}{2(1-\alpha)(\ddot{\psi}-\ddot{\varphi})}. \tag{22}$$

In the next phase we shall need the quantity $\dot{w}(0,\tau_2)$. Since $\dot{\varphi}(\tau_2) = \dot{\psi}(\tau_2)$, then keeping in mind the formulas (18) and (21)–(22) we get:

$$\dot{w}(0,\tau_2) = \dot{\varphi}(\tau_2) = \dot{\varphi}(\tau_1) + \ddot{\varphi}(\tau_2-\tau_1) = \frac{\ddot{\psi}}{2(1-\alpha)(\ddot{\psi}-\ddot{\varphi})}. \tag{23}$$

It is not difficult to calculate the deflections w_2 obtained in the second phase of motion: by integrating the eqns (18) we get

$$w_2(\xi) = \begin{cases} L[(2-\alpha-\xi)\ddot{\psi} - (1-\alpha)\ddot{\varphi} & \text{for} \quad \xi \in [0,\alpha], \\ L(2\ddot{\psi}-\ddot{\varphi})(1-\xi) & \text{for} \quad \xi \in [\alpha,1], \end{cases} \tag{24}$$

where

$$L = 0.125[(1-\alpha)(\ddot{\psi}-\ddot{\varphi})]^{-2}.$$

The third phase

Here we shall express the velocity field in the form

$$\dot{w} = v(t)(1-\xi). \tag{25}$$

The equations of motion (4) we shall integrate for boundary and continuity conditions $Q(0) = 0$, $Q(\alpha-) = Q(\alpha+)$, $M(0) = \gamma^2$, $M(\alpha-) = M(\alpha+)$, $M(1) = 0$. As a result of these calculations, we obtain

$$\dot{v} = -\frac{\gamma^2}{2\Delta\mu},$$

where

$$\mu = \gamma - (\gamma-1)(1-\alpha)^3. \tag{26}$$

Integrating this equation for the conditions $v(\tau_2) = \dot{\varphi}(\tau_2)$ and $v(\tau_f) = 0$, we find the whole response time

$$\tau_f = \tau_2 + \frac{2\Delta\mu}{\gamma^2}\dot{\varphi}(\tau_2) \tag{27}$$

and the residual deflection, obtained in this phase:

$$w_3(\xi) = \frac{\Delta}{\gamma^2}\dot{\varphi}^2(\tau_2)\mu(1-\xi). \tag{28}$$

The quantity $\dot{\varphi}(\tau_2)$ can be calculated according to (23). The inequalities $|M| \leq \gamma^2$ for $\xi \in [0, \alpha]$ and $|M| \leq 1$ for $\xi \in [\alpha, 1]$ are fulfilled.

5. THIRD CASE

This case is the most complicated one, as to the first phase of motion, described in Section 3, the three following phases will be added.

The second phase

Here, besides the hinge at the cross-section $\xi = \alpha$, a stationary hinge appears at $\xi = \beta_* < \alpha$ and we can describe the velocity field as follows:

$$\dot{w} = \begin{cases} 0.5 & \text{for} \quad \xi \in [0, \beta_*], \\ 0.5 - v(t)\dfrac{\xi - \beta}{\alpha - \beta_*} & \text{for} \quad \xi \in [\beta_*, \alpha], \\ [0.5 - v(t)]\dfrac{1-\xi}{1-\alpha} & \text{for} \quad \xi \in [\alpha, 1]. \end{cases} \tag{29}$$

The boundary and continuity conditions for M and Q are now $Q(\beta_*) = 0$, $Q(\alpha -) = Q(\alpha +)$, $M(\beta_*) = \gamma^2$, $M(1) = 1$, $M(\alpha -) = M(\alpha +)$. By integrating the system (4) and satisfying these conditions we get

$$\dot{v} = \frac{\gamma^2 - 1}{\Delta\gamma(\alpha - \beta_*)^2}, \quad (\gamma^2 - 1)(1 - \alpha)[3\gamma(\alpha - \beta_*) + 2(1 - \alpha)] = \gamma(\alpha - \beta_*)^2. \tag{30}$$

It follows from the second equation of (30) that

$$z = \frac{1}{\alpha - \beta_*} = \frac{1}{4(1 - \alpha)}\left(\sqrt{\left(9\gamma^2 + \frac{8\gamma}{\gamma^2 - 1}\right)} - 3\gamma\right). \tag{31}$$

From (31) the unknown quantity β_* can be evaluated. By integrating the first eqn of (30), we obtain

$$v = \frac{\gamma^2 - 1}{\Delta\gamma(\alpha - \beta_*)^2}(\tau - \tau_1). \tag{32}$$

At the instant $\tau = \tau_2$ the hinge at $\xi = \alpha$ disappears and we have $\dot{w}(\alpha -, \tau_2) = \dot{w}(\alpha +, \tau_2)$. In view of (29) and (32) the last condition gives

$$\tau_2 = \tau_1 + \frac{\Delta\gamma(\alpha - \beta_*)^3}{2(\gamma^2 - 1)(1 - \beta_*)}. \tag{33}$$

The resisual deflections, obtained in this phase, will be calculated by means of eqns (29), (32) and (33). Denoting

$$P = \frac{\Delta\gamma(\alpha - \beta_*)^3}{4(\gamma^2 - 1)(1 - \beta_*)}$$

we get

$$w_2 = \begin{cases} P & \text{for} \quad \xi \in [0, \beta_*], \\ \dfrac{P}{2(1 - \beta_*)}(2 - \beta_* - \xi) & \text{for} \quad \xi \in [\beta_*, \alpha], \\ \dfrac{P(1 - \xi)}{2(1 - \beta_*)(1 - \alpha)}(2 - \beta_* - \alpha) & \text{for} \quad \xi \in [\alpha, 1]. \end{cases} \tag{34}$$

This solution will be valid if

$$\gamma\alpha^2 > (1 - \alpha)(\gamma^2 - 1)[3\alpha\gamma + 2(1 - \alpha)]. \tag{35}$$

The third phase

In this phase the hinge at $\xi = \beta_*$ begins to move towards the centre of the beam. Now we shall assume a velocity field

$$\dot{w} = \begin{cases} 0.5 & \text{for} \quad \xi \in [0, \beta], \\ 0.5\dfrac{1-\xi}{1-\beta(\tau)} & \text{for} \quad \xi \in [\beta, 1]. \end{cases}$$

The integration of the eqns (4) with proper boundary and continuity conditions for Q and M enables to draw up the following differential equation

$$\gamma^2(1-\beta)^2 + \Delta\dot{\beta}[\gamma(1-\beta)^3 - (\gamma-1)(1-\alpha)^3] = 0. \tag{37}$$

We shall integrate this equation taking account of the initial condition $\beta(\tau_2) = \beta_*$; the result is

$$\frac{\gamma^2}{\Delta}(\tau - \tau_2) = (\beta_* - \beta)\left[\frac{\gamma}{2}(2 - \beta - \beta_*) - \frac{(\gamma-1)(1-\alpha)^3}{(1-\beta)(1-\beta_*)}\right]. \tag{38}$$

This phase comes to an end, when the hinge has reached the centre. The condition $\beta(\tau_3) = 0$ gives

$$\tau_3 = \tau_2 + \frac{\Delta\beta_*}{\gamma^2}\left[\frac{\gamma}{2}(2-\beta_*) - \frac{(\gamma-1)(1-\alpha)^3}{1-\beta_*}\right]. \tag{39}$$

Now we have to calculate the residual deflections for this phase. It follows from the eqns (36) and (37) that

$$\frac{\partial w}{\partial \beta} = \begin{cases} f(\beta) & \text{for} \quad \xi \in [0, \beta], \\ \dfrac{1-\xi}{1-\beta} f(\beta) & \text{for} \quad \xi \in [\beta, 1], \end{cases} \tag{40}$$

where

$$f(\beta) = -\frac{\Delta}{2\gamma^2}\left[\gamma(1-\beta) - (\gamma-1)\frac{(1-\alpha)^3}{(1-\beta)^2}\right].$$

For $\xi \in [\beta_*, 1]$ we have

$$w_3(\xi) = \int_{\beta_*}^{0} \frac{\partial w}{\partial \beta}\, \mathrm{d}\beta = \frac{\Delta}{2\gamma^2}\left\{\gamma\beta_* - 0.5(\gamma-1)(1-\alpha)^3\left[\frac{1}{(1-\beta_*)^2} - 1\right]\right\}(1-\xi). \tag{41}$$

Analogically we find for $\xi \in [0, \beta_*]$:

$$w_3(\xi) = -\int_{\beta_*}^{\xi} f(\beta)\,\mathrm{d}\beta - (1-\xi)\int_{\xi}^{0} \frac{f(\beta)}{1-\beta}\mathrm{d}\beta = \frac{\Delta}{4\gamma^2}\left[\gamma(2\beta_* - \beta_*{}^2 - \xi^2) + (\gamma-1)(1-\alpha)^3 \right.$$
$$\left. \times\left(\frac{1}{1-\xi} + 1 - \xi - \frac{2}{1-\beta_*}\right)\right]. \tag{42}$$

The fourth phase

Here we have a stationary hinge at $\xi = 0$. The course of solution is quite analogical to what was said about the third phase of the second case. Therefore let us confine ourselves here only to the basic results. The eqns (25)–(26) will be valid. The whole response time is

$$\tau_f = \tau_3 + \frac{\Delta\mu}{\gamma^2}. \tag{43}$$

The residual deflection for this phase is

$$w_4(\xi) = \frac{\Delta\mu}{4\gamma^2}(1-\xi). \tag{44}$$

6. SOLUTION OF THE OPTIMIZATION PROBLEM

In the optimization problem the objective function can be treated in different ways. In this paper we shall consider only two possibilities:

(i) the residual deflection in the centre of the beam must be minimized;

(ii) the mean deflection must be minimal.

In the first case we shall take for the objective function

$$W = w(0, \tau_f) = \sum_{i=1}^{4} w_i(\xi), \tag{45}$$

where the terms $w_i(\xi)$ will be calculated according to the formulas given in Sections 3–5.

The mean deflection will be defined as follows:

$$W_m = \int_0^1 w(\xi, \tau_f)\,d\xi = \sum_{i=1}^{4} \int_0^1 w_i(\xi)\,d\xi. \tag{46}$$

Let us consider a uniform beam for which $\gamma = 1$. The inequality (35) is satisfied and subsequently the third case will be realized. The second eqn (30) gives $\beta_* = \alpha$. Making use of the eqns (10), (34), (42) and (44) we easily find that

$$W^* = W|_{\gamma=1} = 0.5, \quad W_m^* = W_m|_{\gamma=1} = \frac{7}{24} = 0.292. \tag{47}$$

Now it is time to learn about some results of the calculation. Figure 3 shows the occurring regions of the three cases considered in Sections 3–5. It is evident that cases 1 and 3 dominate; the case 2 is realized only in a narrow zone on the plane (α, γ). Dependence of the dimensionless deflections W and W_m on γ for $\alpha = 0.8$ is shown in Figs. 4–5 by the line 1; the point A corresponds to the minimal value of W or W_m. It follows from the calculations that at a given value of α the minimum of W takes place, when the case 1 passes to case 2 (curve 1 in Fig. 3). The global minimum $W = 0.351$ is realized for values $\alpha = 0.787$, $\gamma = 1.66$; as compared to the beam of constant thickness the value of W is reduced by 30%. In a similar way the mean deflection W_m has a minimum on the separating curve between cases 2 and 3 (curve 2 in Fig. 3). In case of a global minimum we have $\alpha = 0.645$, $\gamma = 1.21$ and $W_m = 0.279$. According to (47) in case of a uniform beam we have $W_m^* = 0.292$ and $W_m/W_m^* = 0.96$; so the reduction in mean deflection is 4%.

7. MODE FORM SOLUTIONS

Now let us approximately solve our problem making use of the mode form solutions. The velocity field can be described again by the eqns (18). As before we shall integrate the eqns (4) and determine the constants of the integration from the conditions $Q(0) = 0$, $M(1) = 0$, $Q(\alpha -) = Q(\alpha +)$, $M(\alpha -) = M(\alpha +)$. Satisfying the inequalities $M(0) \leq \gamma^2$, $M(\alpha) \leq 1$ we obtain

$$\begin{gathered}\alpha^2\gamma(3-\alpha)\ddot{\psi} - [3\alpha\gamma(2-\alpha) + 2(1-\alpha)^2](1-\alpha)\ddot{\varphi} \leq \frac{\gamma^2}{\Delta}, \\ -3\alpha^2\gamma(1-\alpha)\ddot{\psi} - 2(1-\alpha)^2(3\alpha\gamma + 1 - \alpha)\ddot{\varphi} \leq \frac{1}{\Delta}.\end{gathered} \tag{48}$$

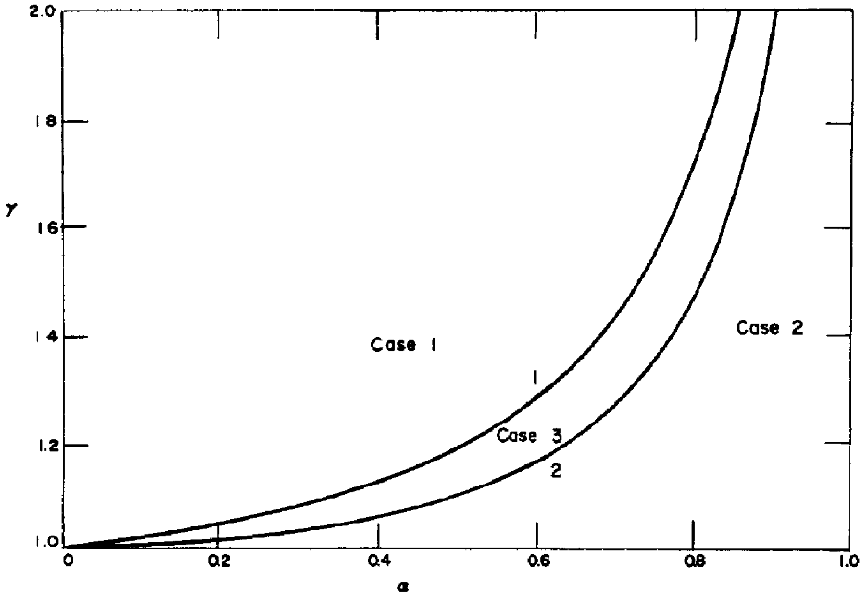

Fig. 3. Regions of occurence of cases 1-3 on the plane (α, γ).

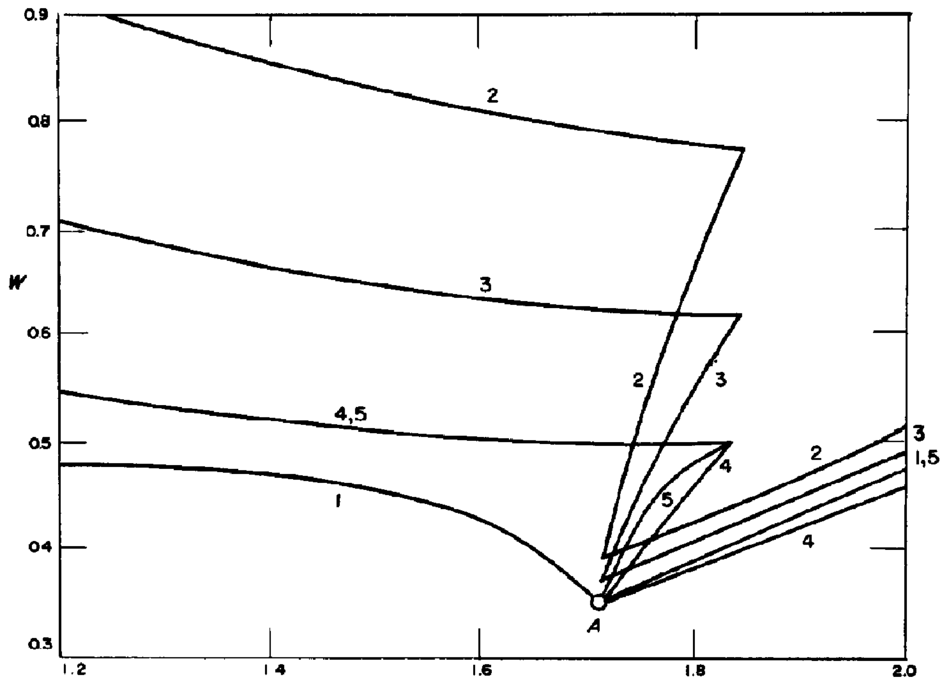

Fig. 4. Dependence of dimensionless central deflection on γ for $\alpha = 0.8$; 1—exact solution, 2-5—mode form solutions.

The following three types of solutions with permanent modes are possible:

(i) A stationary hinge appears at $\xi = \alpha$. In this case $\ddot{\psi} = 0$: the first condition of (48) must be satisfied as strong inequality, the second—as equality.

(ii) We have hinges at $\xi = 0$ and $\xi = \alpha$; both conditions (48) must be satisfied as equalities, besides that $\ddot{\varphi} < \ddot{\psi} < 0$.

(iii) A plastic hinge occurs at the centre $\xi = 0$. Now we have $\ddot{\varphi} = \ddot{\psi}$; the first condition of (48) is satisfied as equality, the second—as strong inequality.

In case of mode form motions

$$\lambda = \frac{\dot{\psi}}{\dot{\varphi}} = \frac{\ddot{\psi}}{\ddot{\varphi}}. \tag{49}$$

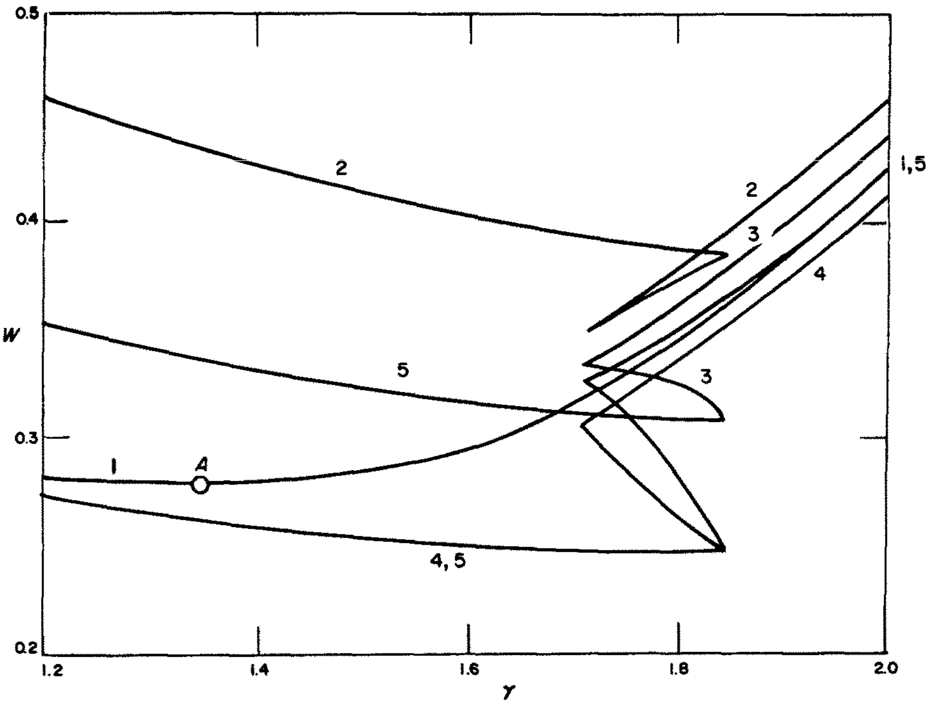

Fig. 5. Dependence of dimensionless mean deflection on γ for $\alpha = 0.8$; 1—exact solution, 2–5—mode form solutions.

Since $\ddot{\varphi} = \text{const.}$, we have $\dot{\varphi} = \dot{\varphi}_0 + \ddot{\varphi}\tau$. Determining the whole response time τ_f from the condition $\dot{\varphi}(\tau_f) = 0$, we get $\tau_f = -\dot{\varphi}_0/\ddot{\varphi}$. The final value of the angle φ is

$$\varphi(\tau_f) = -\frac{1}{2}\frac{\dot{\varphi}_0^2}{\ddot{\varphi}}. \tag{50}$$

An analogical expression also holds good for $\psi(\tau_f)$.

Taking into consideration these results and integrating the eqns (18) we get

$$w(\xi, \tau_f) = \begin{cases} -\dfrac{\dot{\varphi}_0^2}{2\ddot{\varphi}}[1 - \alpha + \lambda(\alpha - \xi)] & \text{for} \quad \xi \in [0, \alpha], \\ -\dfrac{\dot{\varphi}_0^2}{2\ddot{\varphi}}(1 - \xi) & \text{for} \quad \xi \in [\alpha, 1]. \end{cases} \tag{51}$$

The residual deflection in the centre of the beam is

$$w(0, \tau_f) = -\frac{\dot{\varphi}_0^2}{2\ddot{\varphi}}(1 - \alpha + \lambda\alpha). \tag{52}$$

Calculating the mean deflection as given in (46) we obtain

$$w_m(\tau_f) = -\frac{\dot{\varphi}_0^2}{4\ddot{\varphi}}(1 - \alpha^2 + \lambda\alpha^2). \tag{53}$$

Now we have to determine the angular velocity $\dot{\varphi}_0$, for which different possibilities are available. Let us consider some of them.

(1) We may suggest that the moments for the actual initial velocity field $v_0(\xi)$ and for the mode velocity field $\dot{w}(\xi, 0)$ are equal:

$$\int_0^\alpha \gamma v_0(\xi)\mathrm{d}\xi + \int_\alpha^1 v_0(\xi)\,\mathrm{d}\xi = \int_0^\alpha \gamma\dot{w}(\xi, 0)\,\mathrm{d}\xi + \int_\alpha^1 \dot{w}(\xi, 0)\,\mathrm{d}\xi. \tag{54}$$

In this paper we consider only the case $v_0(\xi) = \text{const}$. It was shown in Section 2 that $v_0 \equiv 0.5$. Evaluating the integrals in (54) we find

$$\dot{\varphi} = \frac{\Delta}{(1-\alpha)(2\alpha\gamma + 1 - \alpha) + \gamma\lambda\alpha^2}. \tag{55}$$

(2) The other possibility is to equate the kinetic energies of the velocity fields $v_0(\xi)$ and $\dot{w}(\xi, 0)$. This gives

$$\dot{\varphi}_0{}^2 = \frac{0.75\Delta}{(1-\alpha)^2(3\alpha\gamma + 1 - \alpha) + 3\gamma\lambda\alpha^2(1-\alpha) + \gamma\lambda^2\alpha^3}. \tag{56}$$

This method has been used first by Lippmann[3]. For a stepped beam the initial kinetic energy is $\kappa_0 = \rho B l h_2 \Delta v_0{}^{*2}$. In view of (2) this formula can be put into the form $\kappa_0 = 0.5\rho V v_0{}^{*2}$. Since ρ, V and $v_0{}^*$ are given constants, the quantity κ_0 is also prescribed for all designs. This is the case considered in papers [1, 2].

(3) We may also match the moments-of momenta for one half of the beam with respect to the support $\xi = 1$. In this case we obtain

$$\dot{\varphi} = \frac{1.5[\alpha\gamma(2-\alpha) + (1-\alpha)^2]}{3\alpha\gamma(1-\alpha)(2-\alpha) + 2(1-\alpha)^3 + \gamma\lambda\alpha^2(3-\alpha)}. \tag{57}$$

(4) Symonds and Martin[4] have derived a condition on which the modal form solution approximates the exact solution in the best way. In our notations this condition has the form

$$\int_0^\alpha \gamma v_0 \dot{w}(\xi, 0)\,\mathrm{d}\xi + \int_\alpha^1 v_0 \dot{w}(\xi, 0)\,\mathrm{d}\xi = \int_0^\alpha \gamma \dot{w}^2(\xi, 0)\,\mathrm{d}\xi + \int_\alpha^1 \dot{w}^2(\xi, 0)\,\mathrm{d}\xi.$$

By calculating these integrals we get

$$\dot{\varphi}_0 = \frac{0.75[(1-\alpha)(2\alpha\gamma + 1 - \alpha) + \gamma\lambda\alpha^2]}{(1-\alpha)^2(3\alpha\gamma + 1 - \alpha) + 3\gamma\lambda\alpha^2(1-\alpha) + \gamma\lambda^2\alpha^3}. \tag{58}$$

Some computations according to the formulas (55)–(58) were carried out. The results are presented in Figs. 4–6. Numbers 2–5 in these figures correspond to the cases 1–4 which were considered before.

8. DISCUSSION OF NUMERICAL RESULTS

It follows from Figs. 4–5 that all curves corresponding to the mode form solutions have cupidal points, besides there exists a region of parameters where the solution is not unique. For exact solutions we have a smooth curve (curve 1 in Figs. 4–5) and the solution is always unique. From here we can draw the conclusion that non-uniqueness and instability of some modes are not characteristic of the problem considered in this paper; they are caused by the method of mode form motions.

Comparing the curves 1–5 in Figs. 4–5, we see that from among the mode form solutions the best results are given by the method of Symonds–Martin (curve 5). Also good results can be achieved by equating the moment-of-momenta of the actual initial and mode form velocity fields (curve 4). The most inaccurate results are given by the method where the momenta of both velocity fields are equalized (curve 2).

Let us now consider the case, where the central deflection of the beam W is to be minimized. It follows from Fig. 4 that for a fixed α all curves 1–5 have minima for the same value of the parameter γ, besides all these minima are very close to one another. Consequently, in this case it is not essential which of the mode form solutions, considered in Section 7, we shall use.

As to the global minium of the quantity W, the exact solution and the mode form solutions give different values to the parameters α, γ for which this minimum is realized (e.g. when the

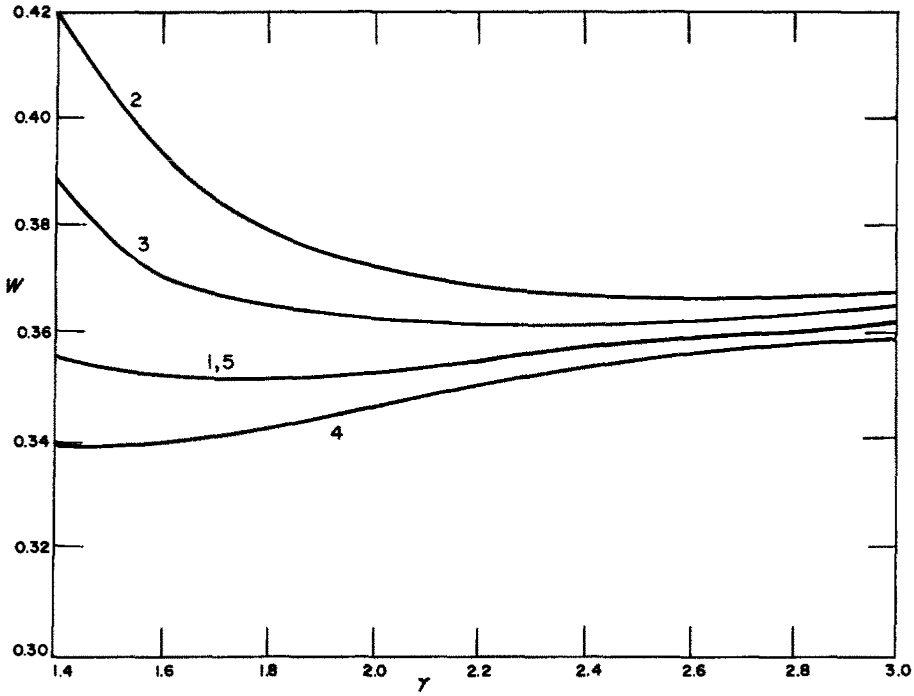

Fig. 6. Dependence of the minimal values of central deflection on γ; 1—exact solution, 2–5—mode form solutions.

initial kinetic energy is specified we have optimal parameters $\alpha = 0.89$, $\gamma = 2.2$, while the exact solution gives $\alpha = 0.79$, $\gamma = 1.7$). At the first glimpse this fact seems to be a serious drawback of the method of modal form solutions. But is follows form the computations that moving along curve 1 in Fig. 3, the value of W does not alter much. To confirm this fact we shall consult Table 1 (the values of α are calculated from (17) substituting an equality sign for the inequality one). Such a conclusion is also valid for mode form solutions (Fig. 6). Hence follows a practical suggestion for optimal design: it is not necessary to find the global minium, but we can use different pairs of parameters α and γ in Table 1. Of course this circumstance gives more freedom to the designer.

Now let us take the mean deflection W_m for the objective function. It follows from Fig. 5 that the minima of this function for the exact solution (curve 1) and for the mode form solutions (curves 2–5) will be at different values of γ. So in this case the mode form solutions do not give good approximations for the real optimal parameters.

9. CONCLUSION

In this paper a problem of optimal design for rigid-plastic beams is exactly solved; these results are compared with four variants of mode form solutions. Resulting from numerical calculations some suggestions for optimal design are made, which can be summed up as follows:

(1) From the mode form solutions, considered before, the best results are given by the solution which is based on the condition of Symonds and Martin[4].

Table 1. Variation of the deflection W along the curve 1 from Fig. 3.

γ	α	W	γ	α	W
1.2	0.524	0.373	2.2	0.887	0.355
1.4	0.681	0.355	2.4	0.906	0.357
1.6	0.768	0.351	2.6	0.921	0.359
1.8	0.823	0.352	2.8	0.932	0.361
2.0	0.860	0.353	3.0	0.942	0.362

(2) If the central deflection of the beam is taken for the objective function, it will be advisable to take the optimal values for parameters α and γ from Table 1.

These conclusions probably also hold good for some more complicated problems. To solve such problems the method of mode form solution may be used, but it would be reasonable to compare these results with exact solutions if they are available.

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